LECTURE 4: FUNDAMENTALS OF GENERAL RELATIVITY (II)

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In this lecture, we follow the convention of Ma & Bertschinger (hereafter M & B 95) [1] to summerise the Friedmann equations for the background and for the perturbations, which is the foundation of the CAMB code. [1]

1. Background

In M & B 95, the conformal time τ was used to denote time, instead of the physical time t, so let us make it clear that we shall use τ from now on, and use the overdot to denote the derivative wrt τ and $'$ for derivative wrt t.

Note that,

$$\tau \equiv \frac{t}{a}; \quad \mathcal{H} \equiv \frac{\mathrm{d}a}{\mathrm{d}\tau}/a = \frac{\dot{a}}{a}; \quad H \equiv \frac{\mathrm{d}a}{\mathrm{d}t}/a = \frac{a'}{a}; \quad \mathcal{H} = aH \tag{1}$$

$$a' = \frac{\dot{a}}{a}, \quad a'' = \frac{1}{a}\left(\frac{\ddot{a}}{a} - \mathcal{H}^2\right) \tag{2}$$

The Friedmann equation in terms of conformal time,

$$\mathcal{H}^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2\rho \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{a}}{a}\right) = -\frac{4\pi G}{3}a^2\left(\rho + 3P\right) \tag{4}$$

2. Perturbation

The general metric including perturbations,

$$\mathrm{d}s^2 = a(\tau)^2\left\{-(1+2\psi)\mathrm{d}\tau^2 + 2w_i\mathrm{d}\tau\mathrm{d}x^i + \left[(1-2\phi)\gamma_{ij} + 2h_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j\right\} \tag{5}$$

where

$$\gamma^{ij}\gamma_{jk} = \delta^i_k, \quad \gamma^{ij}h_{ij} = 0 \tag{6}$$

So there are $1\ (\psi) + 1\ (\phi) + 3\ (w_i) + 6\ (h_{ij}) - 1\ (\gamma^{ij}h_{ij} = 0) = 10$ degrees of freedom, which coincides with the number of independent entries of a 4×4 symmetric metric $g_{\mu\nu}$.

[1] Available at `http://camb.info`

If we focus on scalar perturbations, which is the most important and relevant part for large scale structure of the Universe, only ψ and ϕ are left, in the so-called Conformal-Newton gauge.

In this gauge, different components of the Einstein equation yield,

$$k^2\phi + 3\mathcal{H}(\dot{\phi} + \mathcal{H}\psi) = -4\pi G\rho\delta \tag{7}$$

$$k^2(\dot{\phi} + \mathcal{H}\psi) = 4\pi G a^2(\rho + P)\theta \tag{8}$$

$$\psi = \phi \tag{9}$$

where $\delta \equiv \delta\rho/\rho$, and we have assumed that the fluids are perfect (thus there is no shear perturbations).

On sub-horizon scales, where the density perturbation is much more important than the velocity perturbation, we have the Poision equation, by combining the first two equations,

$$\ddot{\delta} + \mathcal{H}\dot{\delta} = 4\pi G\rho a^2\delta \tag{10}$$

In a matter-dominated Universe, where,

$$H^2 = H_0^2(\Omega_m a^{-3}) \tag{11}$$

so that

$$\mathcal{H}^2 \propto a^{-1} \tag{12}$$

and let us assume,

$$\delta \propto a^n \tag{13}$$

and substitute Eqs (13,12) into Eq (10), we find that,

$$n = 1 \tag{14}$$

This means that in the matter-dominated era, $\delta \propto a$, which is an important conclusion to bear in mind!

An important implication of the Poisson equation Eq (10) is that one can perform consistency tests using it, as $\mathcal{H}$ and $f \equiv \frac{\mathrm{dln}\ \delta}{\mathrm{dln}\ a}$ can be independently measured using SNe and RSD, respectively. If Eq (10) does not hold when the observables are substituted in, it might be a smoking gun of modified gravity!

There are other interesting implications of the Poisson equation. See [2] for an exercise of using it to reconstruct f from $\mathcal{H}$.

References

[1] C. P. Ma and E. Bertschinger, Astrophys. J. **455**, 7 (1995) doi:10.1086/176550 [astro-ph/9506072].

[2] U. Alam, V. Sahni and A. A. Starobinsky, Astrophys. J. **704**, 1086 (2009) doi:10.1088/0004-637X/704/2/1086 [arXiv:0812.2846 [astro-ph]].

[3] Dodelson, S. & Schmidt, F. 2020, Modern cosmology, Academic Press. ISBN 9780128159484, 2020, 512 p.

[4] Peacock, J. A. 1999, Cosmological Physics, by John A. Peacock, pp. 704. ISBN 052141072X. Cambridge, UK: Cambridge University Press, January 1999., 704