

LECTURE 3: FUNDAMENTALS OF GENERAL RELATIVITY (I)

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1. BASICS OF TENSOR ANALYSIS

$$(1) \quad A^\mu \equiv (A^0, A^i), (i = 1, 2, 3)$$

$$(2) \quad A_\mu = g_{\mu\nu} A^\nu; \quad A^\mu = g^{\mu\nu} A_\nu; \quad P_\mu P^\mu = P^2 = g_{\mu\nu} P^\mu P^\nu$$

So the metric can be used to lower or raise indices, even for itself, *e.g.*,

$$(3) \quad g^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta}$$

Set $\alpha = \nu$, we have,

$$(4) \quad g^{\nu\beta} g_{\alpha\beta} = \delta_\alpha^\nu$$

Note that for the FRW metric, the i - i component of $g_{\mu\nu}$ is a^2 , but it is a^{-2} for $g^{\mu\nu}$.

2. THE GEODESIC AND THE AFFINE CONNECTION

The straight line in curves space is called a geodesic, whose equation is,

$$(5) \quad \frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

where the affine connection Γ can be evaluated using the metric, namely,

$$(6) \quad \Gamma_{\alpha\beta}^\mu = \frac{g^{\mu\nu}}{2} \left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right)$$

The connection takes care of the change of the unit vectors when moving from one place to another for an infinitesimal distance on a manifold, so it vanishes if there is no curvature.

Let us calculate the nonzero connections for the FRW metric,

$$(7) \quad \Gamma_{\alpha\beta}^0 = \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial t}$$

So,

$$(8) \quad \Gamma_{00}^0 = \Gamma_{0i}^0 = \Gamma_{i0}^0 = 0$$

$$(9) \quad \Gamma_{ij}^0 = a\dot{a} \delta_{ij}$$

$$(10) \quad \Gamma_{0j}^i = \Gamma_{j0}^i = \frac{\dot{a}}{a} \delta_{ij}$$

3. THE EINSTEIN EQUATION

$$(11) \quad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}; \quad R \equiv g^{\mu\nu}R_{\mu\nu}$$

$$(12) \quad R_{\mu\nu} \equiv \Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\alpha,\nu}^\alpha + \Gamma_{\beta\alpha}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\alpha}^\beta$$

For the background FRW metric,

$$(13) \quad R_{00} = -3\frac{\ddot{a}}{a}; \quad R_{ij} = (2\dot{a}^2 + a\ddot{a}) \delta_{ij}; \quad R = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right]$$

The 0-0 component of the Einstein equation yields,

$$(14) \quad \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho$$

and the i - i component gives us,

$$(15) \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

So the cosmic acceleration ($\ddot{a} > 0$) means that there exists energy component with a negative pressure!

4. ENERGY CONSERVATION

The energy conservation is mathematically equivalent to a vanishing covariant derivative of the energy-momentum tensor, namely,

$$(16) \quad \nabla_\mu T_\nu^\mu = \frac{\partial T_\nu^\mu}{\partial x^\mu} + \Gamma_{\alpha\mu}^\mu T_\nu^\alpha - \Gamma_{\nu\mu}^\alpha T_\alpha^\mu$$

It gives us the continuity equation (set $\nu = 0$), say,

$$(17) \quad \dot{\rho} + 3H\rho(1+w) = 0; \quad w \equiv \frac{P}{\rho}$$

If w is a constant, then,

$$(18) \quad \rho \propto a^{-3(1+w)}$$

So for matter ($w = 0$), radiation ($w = 1/3$) and vacuum energy ($w = -1$), their energy densities evolves with a^{-3} , a^{-4} and a^0 , respectively.

5. HOMEWORK

Ex. 2.2 and 2.6 of Dodelson & Schmidt [1] (page 53-54).

REFERENCES

- [1] Dodelson, S. & Schmidt, F. 2020, Modern cosmology, Academic Press. ISBN 9780128159484, 2020, 512 p.