LECTURE 11: THE MEASUREMENT OF POWER SPECTRUM MULTIPOLES

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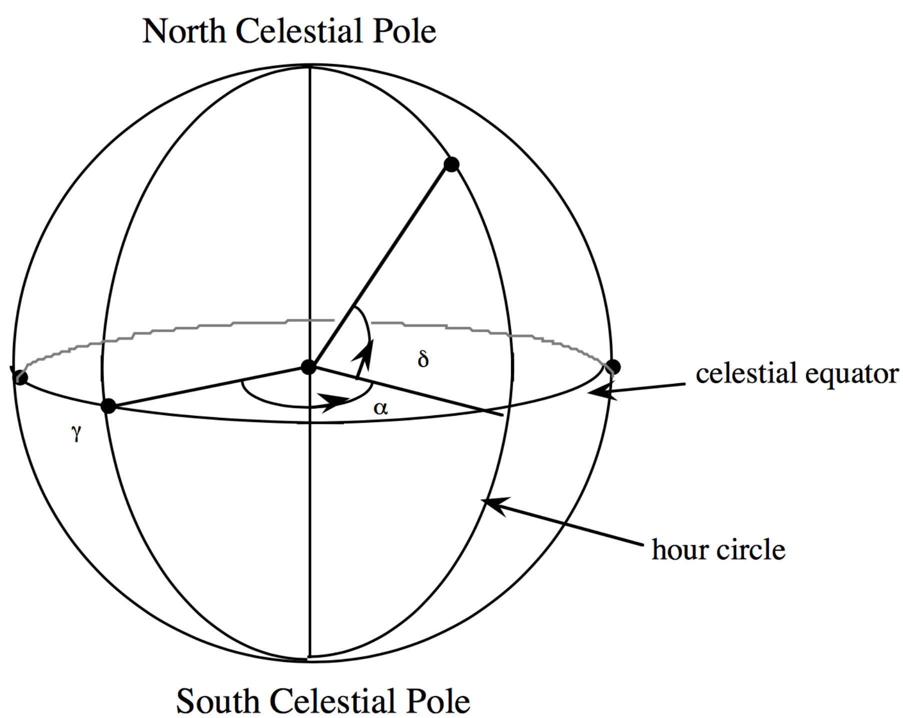
1. Measurement of power spectrum multipoles

$$
\begin{aligned}
x &= \chi \cos\delta \cos\alpha \\
y &= \chi \cos\delta \sin\alpha \\
z &= \chi \sin\delta
\end{aligned} \tag{1}
$$

The Yamamoto estimator,



$$\hat{P}_\ell(k) = \frac{(2\ell+1)}{I}\int\frac{d\Omega_k}{4\pi}\left[\int d\mathbf{r}_1\int d\mathbf{r}_2 F(\mathbf{r}_1)F(\mathbf{r}_2)\times e^{i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}\mathcal{L}_\ell\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}_h\right) - P_\ell^{\rm noise}(\mathbf{k})\right],$$

$$\hat{P}_\ell^{\rm Yama}(k) = \frac{(2\ell+1)}{I}\int\frac{d\Omega_k}{4\pi}\left[\int d\mathbf{r}_1 F(\mathbf{r}_1)e^{i\mathbf{k}\cdot\mathbf{r}_1}\times\int d\mathbf{r}_2 F(\mathbf{r}_2)e^{-i\mathbf{k}\cdot\mathbf{r}_2}\mathcal{L}_\ell\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}_2\right) - P_\ell^{\rm noise}(\mathbf{k})\right],$$

which can not be evaluated using FFTs.

One solution is the following [1],

$$A_n(\mathbf{k}) = \int d\mathbf{r}(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}})^n F(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2}$$

$$\begin{aligned}
\hat{P}_0^{\rm Yama}(k) &= \frac{1}{I}\int\frac{d\Omega_k}{4\pi}\left[A_0(\mathbf{k})A_0^*(\mathbf{k})\right] - P_0^{\rm noise}\\
\hat{P}_2^{\rm Yama}(k) &= \frac{5}{2I}\int\frac{d\Omega_k}{4\pi}A_0(\mathbf{k})\left[3A_2^*(\mathbf{k}) - A_0^*(\mathbf{k})\right]\\
\hat{P}_4^{\rm Yama}(k) &= \frac{9}{8I}\int\frac{d\Omega_k}{4\pi}A_0(\mathbf{k})\left[35A_4^*(\mathbf{k}) - 30A_2^*(\mathbf{k}) + 3A_0^*(\mathbf{k})\right]
\end{aligned} \tag{3}$$

$$\hat{\mathbf{k}}\cdot\hat{\mathbf{r}} = \frac{k_x r_x + k_y r_y + k_z r_z}{kr} \tag{4}$$

$$\begin{aligned}
A_2(\mathbf{k}) =& \frac{1}{k^2}\left\{k_x^2 B_{xx}(\mathbf{k}) + k_y^2 B_{yy}(\mathbf{k}) + k_z^2 B_{zz}(\mathbf{k})\right.\\
&\left.+ 2\left[k_x k_y B_{xy}(\mathbf{k}) + k_x k_z B_{xz}(\mathbf{k}) + k_y k_z B_{yz}(\mathbf{k})\right]\right\}
\end{aligned} \tag{5}$$

$$B_{ij}(\mathbf{k}) \equiv \int d\mathbf{r}\frac{r_i r_j}{r^2}F(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}} \tag{6}$$

$$\begin{aligned}
A_4(\mathbf{k}) =& \frac{1}{k^4}\left\{k_x^4 C_{xxx} + k_y^4 C_{yyy} + k_z^4 C_{zzz}\right.\\
&+ 4\left[k_x^3 k_y C_{xxy} + k_x^3 k_z C_{xxz} + k_y^3 k_x C_{yyx}\right.\\
&\left.+ 4_y^3 k_z C_{yyz} + k_z^3 k_z C_{zzx} + k_z^3 k_y C_{zzy}\right]\\
&+ 6\left[k_x^2 k_y^2 C_{xyy} + k_x^2 k_z^2 C_{xzz} + k_y^2 k_z C_{yzz}\right]\\
&\left.+ 12 k_x k_y k_z\left[k_x C_{xyz} + k_y C_{yxz} + k_z C_{zxy}\right]\right\}
\end{aligned} \tag{7}$$

$$C_{ijl}(\mathbf{k}) \equiv \int d\mathbf{r}\frac{r_i^2 r_j r_l}{r^4}F(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}} \tag{8}$$

This works, but not efficient. A better way is using the addition theorem [2],

$$\mathcal{L}_\ell\left(\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2\right) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}\left(\hat{\mathbf{r}}_1\right) Y_{\ell m}^{\star}\left(\hat{\mathbf{r}}_2\right) \tag{9}$$

which is a generalisation of the well-known formula below,

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \end{aligned} \tag{10}$$

Now,

$$\widehat{P}_\ell(k) = \frac{2\ell+1}{I} \int \frac{\mathrm{d}\Omega_k}{4\pi} F_0(\mathbf{k}) F_\ell(-\mathbf{k}) \tag{11}$$

$$\begin{aligned} F_\ell(\mathbf{k}) &\equiv \int \mathrm{d}\mathbf{r} F(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} \mathcal{L}_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) \\ &= \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{k}}) \int \mathrm{d}\mathbf{r} F(\mathbf{r}) Y_{\ell m}^{*}(\hat{\mathbf{r}}) e^{i\mathbf{k}\cdot\mathbf{r}} \end{aligned} \tag{12}$$

where

$$Y_{\ell m}(\theta, \phi) \equiv \begin{cases} \sqrt{\frac{2\ell+1}{2\pi}\frac{(\ell-m)!}{(\ell+m)!}} \mathcal{L}_\ell^m(\cos\theta)\cos m\phi & m > 0 \\ \sqrt{\frac{2\ell+1}{4\pi}} \mathcal{L}_\ell^m(\cos\theta) & m = 0 \\ \sqrt{\frac{2\ell+1}{2\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}} \mathcal{L}_\ell^{|m|}(\cos\theta)\sin|m|\phi & m < 0 \end{cases} \tag{13}$$

References

[1] D. Bianchi, H. Gil-Marín, R. Ruggeri and W. J. Percival, Mon. Not. Roy. Astron. Soc. **453**, no. 1, L11 (2015) doi:10.1093/mnrasl/slv090 [arXiv:1505.05341 [astro-ph.CO]].

[2] N. Hand, Y. Li, Z. Slepian and U. Seljak, JCAP **1707**, no. 07, 002 (2017) doi:10.1088/1475-7516/2017/07/002 [arXiv:1704.02357 [astro-ph.CO]].